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Perturbation Technique for Non-Self-Adjoint Systems with Repeated Eigenvalues

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The reanalysis of non-self-adjoint systems is computationally very expensive in the design of structural dynamics. An effective, even if approximate, evaluation of the eigensolutions of a modified system is valuable to designers. In a paper by Meirovitch and Ryland (Meirovitch, L., and Ryland, G., "A Perturbation Technique for Gyroscopic Systems with Small Internal and External Damping," *Journal of Sound and Vibration*, Vol. 100, No. 3, 1985, pp. 393–408), a perturbation technique was used to derive the perturbed solutions of a damped gyroscopic system with distinct eigenvalues. That approach is extended to the case of systems with repeated eigenvalues. A numerical example is presented to verify the present method, and satisfactory results are observed.

Nomenclature

A	= $m \times m$ matrix; see Eq. (34)
$\bar{C}$	= $n \times n$ real symmetric damping matrix; see Eq. (1)
c_{ij}	= complex coefficients to be determined
d_{ij}	= complex coefficients to be determined
Ex_i, Ey_i, $E\lambda_i$	= relative errors; see Eqs. (39)
$\bar{G}$	= $n \times n$ skew symmetric gyroscopic matrix; see Eq. (1)
$\bar{H}$	= $n \times n$ skew symmetric circulatory matrix; see Eq. (1)
I	= identity matrix
K, K_0, K_1	= $2n \times 2n$ real general matrices; see Eqs. (3) and (7a)
$\bar{K}$	= $n \times n$ real symmetric stiffness matrix; see Eq. (1)
M, M_0, M_1	= $2n \times 2n$ real general matrices; see Eqs. (3) and (7b)
$\bar{M}$	= $n \times n$ real symmetric mass matrix; see Eq. (1)
m	= multiplicity of eigenvalues
n	= number of degrees of freedom for a system
P	= $m \times m$ matrix; see Eq. (34)
$\boldsymbol{p}^{(k)}$	= kth vectors to be determined; see Eq. (29)
Q	= $m \times m$ matrix; see Eq. (34)
$\boldsymbol{q}^{(k)}$	= kth vectors to be determined; see Eq. (29)
$\boldsymbol{x}_i$	= ith right eigenvector
$\boldsymbol{x}_0^{(k)}$	= kth vectors to be determined; see Eq. (28b)
$\boldsymbol{y}_i$	= ith left eigenvector
$\boldsymbol{y}_0^{(k)}$	= kth vectors to be determined; see Eq. (28a)
$\Delta\boldsymbol{x}_i$, $\Delta\boldsymbol{y}_i$, $\Delta\lambda_i$	= differences between the eigensolutions; see Eqs. (38)
δ_{ij}	= Kronecker delta
ε	= small parameter; see Eqs. (7)
Λ	= $m \times m$ matrix; see Eq. (34)
λ_i	= ith eigenvalue
ϕ_x, ϕ_y	= subspace matrices; see Eqs. (29)
$\lvert \cdot \rvert$	= modulus of a complex number

Subscripts

0, 1, 2	= corresponding to the original system and the first- and second-order perturbations, respectively

Superscripts

H	= conjugate transpose
m_i	= index indicating the element of x_{i0} with the largest modulus
T	= transpose

Introduction

IN the dynamic design of structural systems, a so-called iterative design/analysis process is generally carried out until a satisfactory outcome is achieved to satisfy a variety of requirements of the structural dynamics. Because a full dynamic analysis for a structure is expensive and time consuming, investigators welcome methods that enable a quick evaluation of the changes in the dynamic characteristics as a result of the changes in the system parameters, which may be due to design modifications or improved knowledge of the system. The perturbation technique is one of the most powerful tools that allow one to obtain the results of the modified structures without having to repeat the entire cycle of the dynamic analysis if the changes are small.

For self-adjoint systems, several researchers have worked on the perturbation analysis of a modified design.[1–4] Now perturbation techniques are well developed for systems with distinct, repeated, or closely spaced real eigenvalues. However, many systems give rise to general non-self-adjoint formulations. Important examples are aeroelastic stability of structures and arbitrarily damped or gyroscopic systems. Non-self-adjoint systems differ from self-adjoint systems in two ways, namely, they cannot be described by a single real symmetric matrix, which implies that the eigensolutions are complex for the most part, and solutions of the equations of motion require a state-space description, where the left and right eigenvectors (usually complex) are needed for the so-called decoupling. Courant and Hilbert[5] first studied the perturbation of eigensolutions of a general matrix. Meirovitch and Ryland investigated the perturbation technique for gyroscopic systems with small internal damping[6] and with external damping.[7] Meirovitch[8] studied the perturbation of the eigenvalue problem for general matrices. These studies are limited to the case of distinct complex eigenvalues. However, the situation of repeated complex eigenvalues quite frequently occurs in those engineering structures whose structural stiffness and mass are the same in two perpendicular coordinate directions, e.g., circular cross-sectional shafts, three-dimensional symmetric structures, etc. On the other hand, in some sense, there exists an imperfection in the derivation of two coefficients in the complex expansions adopted in Refs. 5–8. This will be demonstrated and improved in the subsequent analysis.

To this end, in the present paper, extensions are made to the perturbation procedure for non-self-adjoint systems. These allow the computation of perturbed eigensolutions for the systems with repeated complex eigenvalues. The derivations follow the lines of those of Refs. 5 and 8.

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Fundamental Equations

The free vibration equations of motion for a general non-self-adjoint n-degree-of-freedom system can be written in the matrix form[7,8]

$$\bar{M}\ddot{\boldsymbol{q}} + (\bar{C} + \bar{G})\dot{\boldsymbol{q}} + (\bar{K} + \bar{H})\boldsymbol{q} = 0 \tag{1}$$

where $\boldsymbol{q}$ is the $n \times 1$ column vector of generalized coordinates.

To explore the system characteristics, it is necessary to express Eq. (1) in state-space form. Of course, there are special cases of non-self-adjoint systems that can be treated in the configuration space by the classical modal analysis, but here the interest lies in systems that do not lend themselves to such a treatment. To this end, we adjoin the identity $\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}$ and rewrite Eq. (1) in the following state form:

$$K_0\boldsymbol{x} = M_0\dot{\boldsymbol{x}} \tag{2}$$

where $\boldsymbol{x} = [\dot{\boldsymbol{q}}^T, \boldsymbol{q}^T]^T$ is the $2n$-dimensional state vector and

$$K_0 = \begin{bmatrix} -(\bar{C} + \bar{G}) & -(\bar{K} + \bar{H}) \\ I & 0 \end{bmatrix}, \qquad M_0 = \begin{bmatrix} \bar{M} & 0 \\ 0 & I \end{bmatrix} \tag{3a}$$

or

$$K_0 = \begin{bmatrix} \bar{M} & 0 \\ 0 & -(\bar{K} + \bar{H}) \end{bmatrix}, \qquad M_0 = \begin{bmatrix} 0 & \bar{M} \\ \bar{M} & \bar{C} + \bar{G} \end{bmatrix} \tag{3b}$$

The right and left eigenvalue problems of the original (unperturbed) system represented by Eq. (2) are

$$K_0\boldsymbol{x}_{i0} = \lambda_{i0}M_0\boldsymbol{x}_{i0}, \qquad i = 1, 2, \ldots, 2n \tag{4a}$$

$$K_0^T\boldsymbol{y}_{i0} = \lambda_{i0}M_0^T\boldsymbol{y}_{i0}, \qquad i = 1, 2, \ldots, 2n \tag{4b}$$

The left and right eigenvectors are biorthogonal and can be normalized so as to satisfy

$$\boldsymbol{y}_{i0}^T M_0 \boldsymbol{x}_{j0} = \delta_{ij}, \qquad i, j = 1, 2, \ldots, 2n \tag{5}$$

Obviously, Eq. (5) does not render the eigenvectors $\boldsymbol{x}_{i0}$ and $\boldsymbol{y}_{i0}$ uniquely. It must be emphasized that, if the eigenvectors are not unique, neither are their perturbations. Therefore, in this respect, there is an imperfection in Refs. 5–8 because two coefficients in modal expansions are not uniquely determined. To avoid this weakness, a normalization condition has to be imposed to obtain unique eigenvectors. Murthy and Haftka[9] presented a valuable discussion concerning this subject. Now let us consider a normalizing condition

$$\boldsymbol{x}_{i0}^{(m_i)} = 1, \qquad i = 1, 2, \ldots, 2n \tag{6}$$

where the index m_i can be chosen such that $|\boldsymbol{x}_{i0}^{(m_i)}| = \max_{j=1\sim 2n}|\boldsymbol{x}_{i0}^{(j)}|$ and $\boldsymbol{x}_{i0}^{(m_i)}$ and $\boldsymbol{x}_{i0}^{(j)}$ are the m_ith and jth elements of $\boldsymbol{x}_{i0}$, respectively.

The design changes in a structural system may be reflected by the changes in the mass, damping, stiffness, or gyroscopic matrices in Eq. (1). Regardless of the reasons, the net effect is that the matrices K_0 and M_0 in Eq. (2) are different from the original ones. Because these changes are usually small compared to the entire system, the two updated matrices relative to those in Eq. (2) can be expressed as

$$K = K_0 + \varepsilon K_1 \tag{7a}$$

$$M = M_0 + \varepsilon M_1 \tag{7b}$$

respectively, where εK_1 and εM_1 are the corresponding changes and are small relative to K_0 and M_0.

By analogy with Eqs. (4) and (5), the perturbed eigenvalue problems have the form

$$K\boldsymbol{x}_i = \lambda_i M\boldsymbol{x}_i \tag{8a}$$

$$K^T\boldsymbol{y}_i = \lambda_i M^T\boldsymbol{y}_i \tag{8b}$$

where λ_i, $\boldsymbol{x}_i$, and $\boldsymbol{y}_i$, $i = 1, 2, \ldots, 2n$, are the perturbed eigenvalues, right eigenvectors, and left eigenvectors, respectively. The corresponding normalization condition and biorthogonality property are

$$\boldsymbol{x}_i^{(m_i)} = 1, \qquad i = 1, 2, \ldots, 2n \tag{9a}$$

$$\boldsymbol{y}_i^T M\boldsymbol{x}_j = \delta_{ij}, \qquad i, j = 1, 2, \ldots, 2n \tag{9b}$$

respectively.

According to the perturbation theory, the perturbed eigensolutions can be expressed as the analytical function of ε:

$$\lambda_i = \lambda_{i0} + \varepsilon\lambda_{i1} + \varepsilon^2\lambda_{i2} + \cdots, \qquad i = 1, 2, \ldots, 2n \tag{10a}$$

$$\boldsymbol{x}_i = \boldsymbol{x}_{i0} + \varepsilon\boldsymbol{x}_{i1} + \varepsilon^2\boldsymbol{x}_{i2} + \cdots, \qquad i = 1, 2, \ldots, 2n \tag{10b}$$

$$\boldsymbol{y}_i = \boldsymbol{y}_{i0} + \varepsilon\boldsymbol{y}_{i1} + \varepsilon^2\boldsymbol{y}_{i2} + \cdots, \qquad i = 1, 2, \ldots, 2n \tag{10c}$$

Inserting Eqs. (10) into Eqs. (8), considering Eqs. (7), collecting coefficients of the same power of ε, and ignoring third-order terms in the perturbations, we obtain for function ε^1

$$K_0\boldsymbol{x}_{i1} + K_1\boldsymbol{x}_{i0} = \lambda_{i0}M_0\boldsymbol{x}_{i1} + \lambda_{i0}M_1\boldsymbol{x}_{i0} + \lambda_{i1}M_0\boldsymbol{x}_{i0} \qquad i = 1, 2, \ldots, 2n \tag{11a}$$

$$K_0^T\boldsymbol{y}_{i1} + K_1^T\boldsymbol{y}_{i0} = \lambda_{i0}M_0^T\boldsymbol{y}_{i1} + \lambda_{i0}M_1^T\boldsymbol{y}_{i0} + \lambda_{i1}M_0^T\boldsymbol{y}_{i0} \qquad i = 1, 2, \ldots, 2n \tag{11b}$$

and for function ε^2

$$K_0\boldsymbol{x}_{i2} + K_1\boldsymbol{x}_{i1} = \lambda_{i0}M_0\boldsymbol{x}_{i2} + \lambda_{i0}M_1\boldsymbol{x}_{i1} + \lambda_{i1}M_0\boldsymbol{x}_{i1} + \lambda_{i1}M_1\boldsymbol{x}_{i0} + \lambda_{i2}M_0\boldsymbol{x}_{i0}, \qquad i = 1, 2, \ldots, 2n \tag{12a}$$

$$K_0^T\boldsymbol{y}_{i2} + K_1^T\boldsymbol{y}_{i1} = \lambda_{i0}M_0^T\boldsymbol{y}_{i2} + \lambda_{i0}M_1^T\boldsymbol{y}_{i1} + \lambda_{i1}M_0^T\boldsymbol{y}_{i1} + \lambda_{i1}M_1^T\boldsymbol{y}_{i0} + \lambda_{i2}M_0^T\boldsymbol{y}_{i0}, \qquad i = 1, 2, \ldots, 2n \tag{12b}$$

Introducing Eq. (10b) into Eq. (9a), considering Eq. (6), and neglecting third-order quantities, we have for function ε^1

$$\boldsymbol{x}_{i1}^{(m_i)} = 0, \qquad i = 1, 2, \ldots, 2n \tag{13a}$$

and for function ε^2

$$\boldsymbol{x}_{i2}^{(m_i)} = 0, \qquad i = 1, 2, \ldots, 2n \tag{13b}$$

Substituting Eqs. (10b) and (10c) into Eq. (9b), employing Eq. (5), and neglecting third-order quantities, we have for ε^1

$$\boldsymbol{y}_{i0}^T M_0\boldsymbol{x}_{i1} + \boldsymbol{y}_{i1}^T M_0\boldsymbol{x}_{i0} + \boldsymbol{y}_{i0}^T M_1\boldsymbol{x}_{i0} = 0, \qquad i = 1, 2, \ldots, 2n \tag{14a}$$

and for ε^2

$$\boldsymbol{y}_{i0}^T M_0\boldsymbol{x}_{i2} + \boldsymbol{y}_{i1}^T M_0\boldsymbol{x}_{i1} + \boldsymbol{y}_{i0}^T M_1\boldsymbol{x}_{i1} + \boldsymbol{y}_{i1}^T M_1\boldsymbol{x}_{i0} + \boldsymbol{y}_{i2}^T M_0\boldsymbol{x}_{i0} = 0 \qquad i = 1, 2, \ldots, 2n \tag{14b}$$

Up to this stage, we have obtained all of the necessary fundamental equations for the analysis.

Perturbed Eigensolutions in the Case of Distinct Eigenvalues

Expanding the first-order perturbed eigenvectors in the original/unperturbed eigenvectors as follows:

$$\boldsymbol{x}_{i1} = \sum_{j=1}^{2n} c_{ij1}\boldsymbol{x}_{j0}, \qquad i = 1, 2, \ldots, 2n \tag{15a}$$

$$\boldsymbol{y}_{i1} = \sum_{j=1}^{2n} d_{ij1}\boldsymbol{y}_{j0}, \qquad i = 1, 2, \ldots, 2n \tag{15b}$$

substituting Eq. (15a) into Eq. (11a), premultiplying by $\boldsymbol{y}_{j0}^T$, and using Eq. (5), we have

$$c_{ij1}(\lambda_{j0} - \lambda_{i0}) + \boldsymbol{y}_{j0}^T(K_1 - \lambda_{i0}M_1)\boldsymbol{x}_{i0} = \lambda_{i1}\delta_{ij} \tag{16}$$

Letting $i = j$, we obtain the first-order eigenvalue perturbations

$$\lambda_{i1} = \boldsymbol{y}_{i0}^T (K_1 - \lambda_{i0} M_1) \boldsymbol{x}_{i0}, \qquad i = 1, 2, \ldots, 2n \tag{17}$$

On the other hand, for $i \neq j$, we obtain

$$c_{ij1} = \frac{\boldsymbol{y}_{j0}^T (K_1 - \lambda_{i0} M_1) \boldsymbol{x}_{i0}}{\lambda_{i0} - \lambda_{j0}}, \qquad i, j = 1, 2, \ldots, 2n; i \neq j \tag{18}$$

Inserting Eq. (15b) into Eq. (11b) and postmultiplying the transpose of the sequent equation by $\boldsymbol{x}_{i0}$, we have $d_{ij1}(\lambda_{j0} - \lambda_{i0}) + \boldsymbol{y}_{i0}^T (K_1 - \lambda_{i0} M_1) \boldsymbol{x}_{j0} = \lambda_{i1} \delta_{ij}$. For $i = j$, we once again obtain λ_{i1}; for $i \neq j$, we obtain

$$d_{ij1} = \frac{\boldsymbol{y}_{i0}^T (K_1 - \lambda_{i0} M_1) \boldsymbol{x}_{j0}}{\lambda_{i0} - \lambda_{j0}}, \qquad i, j = 1, 2, \ldots, 2n; i \neq j \tag{19}$$

It is evident that c_{ii1} and d_{ii1} remain unknown. In Refs. 5 and 8, the two coefficients are omitted; in Refs. 6 and 7, they are defined under the assumption of $c_{ii1} = d_{ii1}$. These two treatments are not always mathematically strict. To improve this situation, now let us determine c_{ii1} and d_{ii1} via Eqs. (6) and (9a).

Premultiplying Eq. (15a) by $\boldsymbol{y}_{i0}^T M_0$ and considering Eq. (5), we have

$$c_{ii1} = \boldsymbol{y}_{i0}^T M_0 \boldsymbol{x}_{i1}, \qquad i = 1, 2, \ldots, 2n \tag{20}$$

Postmultiplying the transpose of Eq. (15b) by $M_0 \boldsymbol{x}_{i0}$ and using Eq. (5), we have

$$d_{ii1} = \boldsymbol{y}_{i1}^T M_0 \boldsymbol{x}_{i0}, \qquad i = 1, 2, \ldots, 2n \tag{21}$$

Substituting Eqs. (20) and (21) into Eq. (14a), we obtain

$$c_{ii1} + d_{ii1} = -\boldsymbol{y}_{i0}^T M_1 \boldsymbol{x}_{i0}, \qquad i = 1, 2, \ldots, 2n \tag{22}$$

In view of Eq. (13a) and considering Eq. (6), we have

$$c_{ii1} = -\sum_{j=1, \neq i}^{2n} c_{ij1} \boldsymbol{x}_{j0}^{(m_i)}, \qquad i = 1, 2, \ldots, 2n \tag{23}$$

Substituting Eq. (23) into Eq. (22), we obtain

$$d_{ii1} = \sum_{j=1, \neq i}^{2n} c_{ij1} \boldsymbol{x}_{j0}^{(m_i)} - \boldsymbol{y}_{i0}^T M_1 \boldsymbol{x}_{i0}, \qquad i = 1, 2, \ldots, 2n \tag{24}$$

Synthesizing Eqs. (15), (17–19), (23), and (24), we obtain the whole first-order perturbations as follows:

$$\lambda_{i1} = \boldsymbol{y}_{i0}^T \big(K_1 - \lambda_{i0} M_1\big) \boldsymbol{x}_{i0}, \qquad i = 1, 2, \ldots, 2n \tag{25a}$$

$$\boldsymbol{x}_{i1} = \sum_{j=1, \neq i}^{2n} \frac{\boldsymbol{y}_{j0}^T (K_1 - \lambda_{i0} M_1) \boldsymbol{x}_{i0}}{\lambda_{i0} - \lambda_{j0}} \big(\boldsymbol{x}_{j0} - \boldsymbol{x}_{j0}^{(m_i)} \boldsymbol{x}_{i0}\big) \qquad i = 1, 2, \ldots, 2n \tag{25b}$$

$$\boldsymbol{y}_{i1} = \sum_{j=1, \neq i}^{2n} \frac{\boldsymbol{y}_{i0}^T (K_1 - \lambda_{i0} M_1) \boldsymbol{x}_{j0}}{\lambda_{i0} - \lambda_{j0}} \boldsymbol{y}_{j0} + \left[\sum_{j=1, \neq i}^{2n} \frac{\boldsymbol{y}_{j0}^T (K_1 - \lambda_{i0} M_1) \boldsymbol{x}_{i0}}{\lambda_{i0} - \lambda_{j0}} \boldsymbol{x}_{j0}^{(m_i)} - \boldsymbol{y}_{i0}^T M_1 \boldsymbol{x}_{i0} \right] \boldsymbol{y}_{i0} \qquad i = 1, 2, \ldots, 2n \tag{25c}$$

To compute $\lambda_{i2}, \boldsymbol{x}_{i2}$, and $\boldsymbol{y}_{i2}$, we first expand $\boldsymbol{x}_{i2}$ and $\boldsymbol{y}_{i2}$ in $\boldsymbol{x}_{i0}$ and $\boldsymbol{y}_{i0}$:

$$\boldsymbol{x}_{i2} = \sum_{j=1}^{2n} c_{ij2} \boldsymbol{x}_{j0}, \qquad i = 1, 2, \ldots, 2n \tag{26a}$$

$$\boldsymbol{y}_{i2} = \sum_{j=1}^{2n} d_{ij2} \boldsymbol{y}_{j0}, \qquad i = 1, 2, \ldots, 2n \tag{26b}$$

Using the same pattern and working with Eqs. (12), (13b), (14b), and (26), we conclude that the second-order perturbations have the form

$$\lambda_{i2} = \boldsymbol{y}_{i0}^T (K_1 - \lambda_{i0} M_1 - \lambda_{i1} M_0) \boldsymbol{x}_{i1} - \lambda_{i1} \boldsymbol{y}_{i0}^T M_1 \boldsymbol{x}_{i0} \qquad i = 1, 2, \ldots, 2n \tag{27a}$$

$$\boldsymbol{x}_{i2} = \sum_{j=1, \neq i}^{2n} \frac{\boldsymbol{y}_{j0}^T [(K_1 - \lambda_{i0} M_1 - \lambda_{i1} M_0) \boldsymbol{x}_{i1} - \lambda_{i1} M_1 \boldsymbol{x}_{i0}]}{\lambda_{i0} - \lambda_{j0}} \times \big(\boldsymbol{x}_{j0} - \boldsymbol{x}_{j0}^{(m_i)} \boldsymbol{x}_{i0}\big), \qquad i = 1, 2, \ldots, 2n \tag{27b}$$

$$\boldsymbol{y}_{i2} = \sum_{j=1, \neq i}^{2n} \frac{\boldsymbol{y}_{i1}^T (K_1 - \lambda_{i0} M_1 - \lambda_{i1} M_0) \boldsymbol{x}_{j0} - \lambda_{i1} \boldsymbol{y}_{i0}^T M_1 \boldsymbol{x}_{j0}}{\lambda_{i0} - \lambda_{j0}} \boldsymbol{y}_{j0} + \sum_{j=1, \neq i}^{2n} \frac{\boldsymbol{y}_{j0}^T [(K_1 - \lambda_{i0} M_1 - \lambda_{i1} M_0) \boldsymbol{x}_{i1} - \lambda_{i1} M_1 \boldsymbol{x}_{i0}]}{\lambda_{i0} - \lambda_{j0}} \boldsymbol{x}_{j0}^{(m_i)} \boldsymbol{y}_{i0} - \big(\boldsymbol{y}_{i1}^T M_0 \boldsymbol{x}_{i1} + \boldsymbol{y}_{i0}^T M_1 \boldsymbol{x}_{i1} + \boldsymbol{y}_{i1}^T M_1 \boldsymbol{x}_{i0}\big) \boldsymbol{y}_{i0} \qquad i = 1, 2, \ldots, 2n \tag{27c}$$

Perturbed Eigensolutions in the Case of Repeated Eigenvalues

When the solution of Eqs. (4) produces m repeated eigenvalues, i.e., $\lambda_{i0} = \lambda_{(i+1,0)} = \cdots = \lambda_{(i+m-1,0)} \Rightarrow \lambda_0$, then computation of perturbations of the eigensolutions is not straightforward. The complication is related to the left and right eigenvectors corresponding to the repeated eigenvalues not being unique. In fact, an infinite number of linear combinations of the eigenvectors, e.g.,

$$\boldsymbol{y}_0^{(k)} = \sum_{j=i}^{i+m-1} \boldsymbol{p}_j^{(k)} \boldsymbol{y}_{j0} = \phi_y \boldsymbol{p}^{(k)} \tag{28a}$$

$$\boldsymbol{x}_0^{(k)} = \sum_{j=i}^{i+m-1} \boldsymbol{q}_j^{(k)} \boldsymbol{x}_{j0} = \phi_x \boldsymbol{q}^{(k)} \tag{28b}$$

will satisfy Eqs. (4), where $k = 1, 2, \ldots, \infty$; coefficients $\boldsymbol{p}_j^{(k)}$ and $\boldsymbol{q}_j^{(k)}$ are to be determined and

$$\begin{aligned} \phi_y &= \big[\boldsymbol{y}_{i0}, \boldsymbol{y}_{(i+1,0)}, \ldots, \boldsymbol{y}_{(i+m-1,0)}\big] \\ \boldsymbol{p}^{(k)} &= \big[\boldsymbol{p}_i^{(k)}, \boldsymbol{p}_{i+1}^{(k)}, \ldots, \boldsymbol{p}_{i+m-1}^{(k)}\big]^T \\ \phi_x &= \big[\boldsymbol{x}_{i0}, \boldsymbol{x}_{(i+1,0)}, \ldots, \boldsymbol{x}_{(i+m-1,0)}\big] \\ \boldsymbol{q}^{(k)} &= \big[\boldsymbol{q}_i^{(k)}, \boldsymbol{q}_{i+1}^{(k)}, \ldots, \boldsymbol{q}_{i+m-1}^{(k)}\big]^T \end{aligned} \tag{29}$$

Substitution of $\boldsymbol{y}_0^{(k)}$ and $\boldsymbol{x}_0^{(k)}$ in Eqs. (28) for $\boldsymbol{y}_{i0}$ and $\boldsymbol{x}_{i0}$ in Eqs. (10) and consideration of alternating the associated superscript and subscript indices yields

$$\lambda_j = \lambda_{j0} + \varepsilon \lambda_{j1} + \varepsilon^2 \lambda_{j2} + \cdots, \qquad j = i, i+1, \ldots, i+m-1 \tag{30a}$$

$$\boldsymbol{x}_j = \boldsymbol{x}_0^{(j)} + \varepsilon \boldsymbol{x}_{j1} + \varepsilon^2 \boldsymbol{x}_{j2} + \cdots, \qquad j = i, i+1, \ldots, i+m-1 \tag{30b}$$

$$\boldsymbol{y}_j = \boldsymbol{y}_0^{(j)} + \varepsilon \boldsymbol{y}_{j1} + \varepsilon^2 \boldsymbol{y}_{j2} + \cdots, \qquad j = i, i+1, \ldots, i+m-1 \tag{30c}$$

Comparison of Eqs. (30b) and (30c) with Eqs. (10b) and (10c) reveals that the only difference lies in the first terms on the right-hand

sides. Substituting Eqs. (15) into Eq. (11a), considering the alternations of the associated indices, and premultiplying the sequent equation by $\boldsymbol{y}_{k0}^T$, we obtain

$$\boldsymbol{y}_{k0}^T(K_0-\lambda_{j0}M_0)\sum_{l=1}^{2n}c_{jl1}\boldsymbol{x}_{l0}+\boldsymbol{y}_{k0}^T(K_1-\lambda_{j0}M_1)\phi_x\boldsymbol{q}^{(j)}$$
$$=\lambda_{j1}\boldsymbol{y}_{k0}^TM_0\phi_x\boldsymbol{q}^{(j)}$$
$$j=i,i+1,\ldots,i+m-1,\quad k=1,2,\ldots,2n \tag{31}$$

By considering Eqs. (5) and (28b), Eq. (31) is changed to a new m-order standard eigenvalue problem:

$$AQ=Q\Lambda \tag{32}$$

Similarly, working with Eqs. (5), (11b), (15), and (28a), we obtain the adjoint eigenvalue problem of Eq. (32):

$$A^TP=P\Lambda \tag{33}$$

where

$$A=\phi_y^T(K_1-\lambda_0M_1)\phi_x,\qquad P=\left[\boldsymbol{p}^{(i)},\boldsymbol{p}^{(i+1)},\ldots,\boldsymbol{p}^{(i+m-1)}\right]$$
$$Q=\left[\boldsymbol{q}^{(i)},\boldsymbol{q}^{(i+1)},\ldots,\boldsymbol{q}^{(i+m-1)}\right] \tag{34}$$
$$\Lambda=\mathrm{diag}\left[\lambda_{i1},\lambda_{(i+1,1)},\ldots,\lambda_{(i+m-1,1)}\right]$$

Substituting Eqs. (30b) and (30c) into Eqs. (9) and considering Eqs. (5) and (6), we have

$$\left(\boldsymbol{q}^{(j)}\right)^T\phi_x^T\phi_x\boldsymbol{q}^{(j)}=1,\qquad j=i,i+1,\ldots,i+m-1 \tag{35a}$$

$$\left(\boldsymbol{p}^{(j)}\right)^T\boldsymbol{q}^{(j)}=1,\qquad j=i,i+1,\ldots,i+m-1 \tag{35b}$$

which are the associated normalization conditions corresponding to Eqs. (32) and (33), respectively.

Solving Eqs. (32) and (35a), we obtain λ_{j1} ($j=i,i+1,\ldots,i+m-1$), which are the first-order perturbations of the m repeated eigenvalues. Meanwhile, we obtain the unique m right eigenvectors $\boldsymbol{q}^{(j)}$, i.e., the m coefficient vectors to be determined. Solving Eqs. (33) and (35b), we obtain λ_{j1} once more and the unique m left eigenvectors $\boldsymbol{p}^{(j)}$. In fact, considering Eqs. (28), we uniquely determine the eigenvectors $\boldsymbol{x}_0^{(j)}$ and $\boldsymbol{y}_0^{(j)}$ in Eqs. (30) corresponding to the repeated eigenvalues in the unperturbed system.

As was already pointed out, Eqs. (30) are the same as Eqs. (10) in form. Therefore, with the knowledge of $\boldsymbol{x}_0^{(j)}$ and $\boldsymbol{y}_0^{(j)}$ in Eqs. (30), the same solution procedure can be performed as discussed in the preceding section. In the perturbed eigensolutions for distinct eigenvalues, i.e., in Eqs. (25) and (27), if $j,k=i,i+1,\ldots,i+m-1$, only two substitutions are required:

$$\boldsymbol{x}_{j0}\,(\text{or}\,\boldsymbol{x}_{k0})\to\boldsymbol{x}_0^{(j)}\left(\text{or}\,\boldsymbol{x}_0^{(k)}\right),\qquad \boldsymbol{y}_{j0}\,(\text{or}\,\boldsymbol{y}_{k0})\to\boldsymbol{y}_0^{(j)}\left(\text{or}\,\boldsymbol{y}_0^{(k)}\right) \tag{36}$$

thus giving the perturbations for the case of repeated eigenvalues. Moreover, at this time the notation

$$\sum_{j=1,\neq i}^{2n}$$

in Eqs. (25) and (27) indicates $j=1,2,\ldots,2n$, but $j\neq i,i+1,\ldots,i+m-1$ because of the m repeated eigenvalues.

Illustrative Example

The equations of motion of the system given in Fig. 1 are

$$m_1\ddot{q}_1+c_1\dot{q}_1+(k_1+k)q_1-kq_2=0$$
$$m_2\ddot{q}_2+c_2\dot{q}_2+(k_2+k)q_2-kq_1=0 \tag{37}$$

Table 1 Eigensolutions of the unperturbed system ($k=0$)

λ_{i0}	$\boldsymbol{x}_{i0}$	$\boldsymbol{y}_{i0}$
$-0.25\pm0.968245i$	$-1.11803\pm0.86603i$	$-0.22361\mp0.28868i$
	$1.11803\pm0.86603i$	$0.22361\mp0.28868i$
	0	0
	0	0
$-0.25\pm0.968245i$	0	0
	0	0
	$-1.11803\pm0.86603i$	$-0.22361\mp0.28868i$
	$1.11803\pm0.86603i$	$0.22361\mp0.28868i$

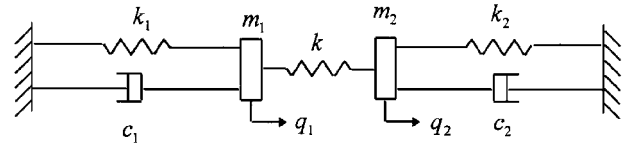

Fig. 1 System with repeated eigenvalues.

Introducing state variables $\boldsymbol{x}=[\dot{q}_1,q_1,\dot{q}_2,q_2]^T$, we have $K\boldsymbol{x}=M\dot{\boldsymbol{x}}$, in which

$$K=\begin{bmatrix}-c_1 & -(k_1+k) & 0 & k\\ 1 & 0 & 0 & 0\\ 0 & k & -c_2 & -(k_2+k)\\ 0 & 0 & 1 & 0\end{bmatrix}$$

$$M=\begin{bmatrix}m_1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & m_2 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Appointing the system of $k=0$ to be the unperturbed one, we have

$$K_0=\begin{bmatrix}-c_1 & -k_1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & k & -c_2 & -k_2\\ 0 & 0 & 1 & 0\end{bmatrix},\qquad \varepsilon K_1=\begin{bmatrix}0 & -k & 0 & k\\ 0 & 0 & 0 & 0\\ 0 & k & 0 & -k\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$M_0=M,\qquad \varepsilon M_1=0$$

Supposing $m_1=m_2=k_1=k_2=1$, $c_1=c_2=0.5$, and $k=0$, the unperturbed eigensolutions obtained by the QR method[8] are listed in Table 1, in which there are two groups of repeated eigenvalues. Letting $k=0.1$, the perturbed eigensolutions are listed in Table 2. Note that the eigenvectors are normalized by Eqs. (5), (6), and (9), but they are presented in the well-known complex-conjugate form in Tables 1 and 2 through a simple operation. This is merely a matter of the beauty of mathematics.

Table 3 lists the differences between the eigensolutions obtained by the present method (PM) and the QR method. The definition of the differences is

$$\Delta\lambda_i=(\lambda_i)_{\mathrm{PM}}-(\lambda_i)_{\mathrm{QR}} \tag{38a}$$

$$\Delta\boldsymbol{x}_i=(\boldsymbol{x}_i)_{\mathrm{PM}}-(\boldsymbol{x}_i)_{\mathrm{QR}} \tag{38b}$$

$$\Delta\boldsymbol{y}_i=(\boldsymbol{y}_i)_{\mathrm{PM}}-(\boldsymbol{y}_i)_{\mathrm{QR}} \tag{38a}$$

The relative errors of the eigensolutions are defined as

$$E\lambda_i=\frac{|\Delta\lambda_i|}{|(\lambda_i)_{\mathrm{QR}}|}\times100\% \tag{39a}$$

$$Ex_i=\frac{\sqrt{\Delta x_i^H\Delta x_i}}{\sqrt{(x_i)_{\mathrm{QR}}^H(x_i)_{\mathrm{QR}}}}\times100\% \tag{39b}$$

$$Ey_i=\frac{\sqrt{\Delta y_i^H\Delta y_i}}{\sqrt{(y_i)_{\mathrm{QR}}^H(y_i)_{\mathrm{QR}}}}\times100\% \tag{39c}$$

The relative errors are also listed in Table 3 in parentheses.

Table 2 Eigensolutions of the perturbed system ($k = 0.1$)

Method	λ_i	x_i	y_i
QR	$-0.25 \pm 0.96825i$	$-0.79057 \pm 0.61237i$	$-0.15811 \mp 0.20412i$
		$0.79057 \pm 0.61237i$	$0.15811 \mp 0.20412i$
		$-0.79057 \pm 0.61237i$	$-0.15811 \mp 0.20412i$
		$0.79057 \pm 0.61237i$	$0.15811 \mp 0.20412i$
	$-0.25 \pm 1.0665i$	$0.93367 \mp 0.49237i$	$0.18360 \pm 0.15959i$
		$-0.63213 \mp 0.72725i$	$-0.12431 \pm 0.23571i$
		$-0.93367 \pm 0.49237i$	$-0.18360 \mp 0.15959i$
		$0.63213 \pm 0.72725i$	$0.12431 \mp 0.23571i$
Present (first order)	$-0.25 \pm 0.96825i$	$-0.79057 \pm 0.61237i$	$-0.15811 \mp 0.20412i$
		$0.79057 \pm 0.61237i$	$0.15811 \mp 0.20412i$
		$-0.79057 \pm 0.61237i$	$-0.15811 \mp 0.20412i$
		$0.79057 \pm 0.61237i$	$0.15811 \mp 0.20412i$
	$-0.25 \pm 1.0715i$	$0.98031 \mp 0.53072i$	$0.18341 \pm 0.14969i$
		$-0.66408 \mp 0.77567i$	$-0.12017 \pm 0.23134i$
		$-0.98031 \pm 0.53072i$	$-0.18341 \mp 0.14969i$
		$0.66408 \pm 0.77567i$	$0.12017 \mp 0.23134i$
Present (second order)	$-0.25 \pm 0.96825i$	$-0.79057 \pm 0.61237i$	$-0.15811 \mp 0.20412i$
		$0.79057 \pm 0.61237i$	$0.15811 \mp 0.20412i$
		$-0.79057 \pm 0.61237i$	$-0.15811 \mp 0.20412i$
		$0.79057 \pm 0.61237i$	$0.15811 \mp 0.20412i$
	$-0.25 \pm 1.0660i$	$0.94489 \mp 0.47629i$	$0.18442 \pm 0.16058i$
		$-0.61601 \mp 0.73757i$	$-0.12371 \pm 0.23678i$
		$-0.94489 \pm 0.47629i$	$-0.18442 \mp 0.16058i$
		$0.60601 \pm 0.73757i$	$0.12371 \mp 0.23678i$

Table 3 Errors of the perturbed eigensolutions ($k = 0.1$)

Order	$\Delta\lambda_i$ $(E\lambda_i)$	Δx_i (Ex_i)	Δy_i (Ey_i)
First	0	0	0
	(0)	0	0
		0	0
		0	0
		(0)	(0)
	$\pm 0.005i$	$0.04664 \mp 0.03835i$	$-0.00019 \mp 0.00990i$
	(0.41669%)	$-0.03195 \mp 0.04842i$	$0.00414 \mp 0.00437i$
		$-0.04664 \pm 0.03835i$	$0.00019 \pm 0.00990i$
		$0.03195 \pm 0.04842i$	$-0.00414 \pm 0.00437i$
		(5.8587%)	(3.4324%)
Second	0	0	0
	(0)	0	0
		0	0
		0	0
		(0)	(0)
	$\mp 0.0005i$	$0.01122 \pm 0.01608i$	$0.00082 \pm 0.00099i$
	(0.041669%)	$0.01612 \mp 0.01032i$	$0.00060 \pm 0.00107i$
		$-0.01122 \mp 0.01608i$	$-0.00082 \mp 0.00099i$
		$-0.01612 \pm 0.01032i$	$-0.00060 \mp 0.00107i$
		(1.9172%)	(0.49247%)

There are two problems not discussed in the paper but that are worthy of consideration. For many systems, especially for large complicated structures, quite often it is difficult to obtain all of the eigensolutions. In this situation, truncated modal analysis[10] may need to be performed. On the other hand, some of the non-self-adjoint systems with repeated eigenvalues may be defective, which requires a more thorough investigation.

Concluding Remarks

From the results as shown in Table 3, it can be observed that, in the case of repeated eigenvalues, the first-order perturbed eigensolutions obtained by the present method have sufficient precision compared with the numerical solutions calculated by the QR method and that the second-order approximations are nearly equal to the QR solutions. Therefore, it can be predicted that the present method will very likely give calculated results sufficiently accurate for engineering purposes in the case of large complicated structures.

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References

[1]Chen, J. C., and Wada, B. K., "Matrix Perturbation for Structural Dynamic Analysis," *AIAA Journal*, Vol. 15, No. 8, 1977, pp. 1095–1100.

[2]Baldwin, J. F., and Hutton, S. G., "Natural Modes of Modified Structures," *AIAA Journal*, Vol. 23, No. 11, 1985, pp. 1737–1743.

[3]Chen, S. H., Liu, Z. S., Shao, C. S., and Zhao, Y. Q., "Perturbation Analysis of Vibration Modes with Close Frequencies," *Communications in Numerical Methods in Engineering*, Vol. 9, No. 6, 1993, pp. 427–438.

[4]Chen, J. Y., Liu, J. K., and Zhao, L. C., "An Improved Perturbation Method for Free Vibration Analysis," *Journal of Sound and Vibration*, Vol. 180, No. 3, 1995, pp. 519–523.

[5]Courant, R., and Hilbert, D., *Methods of Mathematical Physics*, Vol. 1, Interscience, New York, 1953, pp. 343–348.

[6]Meirovitch, L., and Ryland, G., "Response of Lightly Damped Gyroscopic Systems," *Journal of Sound and Vibration*, Vol. 67, No. 1, 1979, pp. 1–19.

[7]Meirovitch, L., and Ryland, G., "A Perturbation Technique for Gyroscopic Systems with Small Internal and External Damping," *Journal of Sound and Vibration*, Vol. 100, No. 3, 1985, pp. 393–408.

[8]Meirovitch, L., *Principles and Techniques of Vibrations*, Prentice–Hall, Englewood Cliffs, NJ, 1997, pp. 259–262, 348–350.

[9]Murthy, D. V., and Haftka, R. T., "Derivatives of Eigenvalues and Eigenvectors of a General Complex," *International Journal for Numerical Methods in Engineering*, Vol. 26, No. 2, 1988, pp. 293–311.

[10]Wang, B. P., "Improved Approximation Methods for Computing Eigenvector Derivatives in Structural Dynamics," *AIAA Journal*, Vol. 29, No. 6, 1991, pp. 1018–1020.

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